

# Nagata embedding and $\mathcal{A}$ -schemes

Satoshi Takagi

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## Abstract

We define the notion of normal  $\mathcal{A}$ -schemes, and approximable  $\mathcal{A}$ -schemes. Approximable  $\mathcal{A}$ -schemes inherit many good properties of ordinary schemes. As a consequence, we see that the Zariski-Riemann space can be regarded in two ways – either as the limit space of admissible blow ups, or as the universal compactification of a given non-proper scheme. We can prove Nagata embedding using Zariski-Riemann spaces.

## Contents

|          |   |           |
|----------|---|-----------|
| <b>0</b> | <b>Introduction</b>                                       | <b>1</b>  |
| <b>1</b> | <b>A brief review of <math>\mathcal{A}</math>-schemes</b> | <b>3</b>  |
| <b>2</b> | <b>Normalization</b>                                      | <b>5</b>  |
| <b>3</b> | <b>Approximations by ordinary schemes</b>                 | <b>8</b>  |
| <b>4</b> | <b>Another proof of Nagata embedding</b>                  | <b>11</b> |

## 0 Introduction

We introduced the concept of  $\mathcal{A}$ -schemes in [T]. In this paper, we will investigate further properties of  $\mathcal{A}$ -schemes, mainly focusing on Zariski-Riemann spaces.

First, we will show that there is a normalization of  $\mathcal{A}$ -schemes, just as for ordinary schemes. This is important, since we are aiming for an analog of Zariski's main theorem.

One of the advantage of introducing  $\mathcal{A}$ -schemes is that we can simplify the proof of Nagata embedding theorem; it can be proven intuitively, as in the original paper of Nagata [N1]. Note that the essential part is already proven in Corollary 4.4.6 of [T]. Compare with the proof of Conrad [C], which only uses ordinary schemes, but is long (approximately 50 pages).

Also, we introduce the notion of approximable  $\mathcal{A}$ -schemes: an  $\mathcal{A}$ -scheme is approximable if it is a (filtered) projective limit of ordinary schemes. This notion is convenient, since locally free sheaves on approximable schemes always come from a pull back of a locally free sheaves on ordinary schemes. At the same time, we see that the Zariski-Riemann space defined in [T] is identified with the conventional one, namely the limit space of  $U$ -admissible blow ups along the exceptional locus  $X \setminus U$  where  $U$  is an open subscheme of a scheme  $X$ . This shows that the conventional Zariski-Riemann space has the desirable universal property in the category of  $\mathcal{A}$ -schemes, not only with schemes.

This paper is organized as follows. In section 1, we quickly review the definitions and properties of  $\mathcal{A}$ -schemes, which plays the central role in this paper. In section 2, we construct the normalization functor of  $\mathcal{A}$ -schemes. In section 3, we give a notion of approximable  $\mathcal{A}$ -schemes. This actually determines the complete hull of the category  $(\mathcal{Q}\text{-}\mathbf{Sch})$  of ordinary schemes in the category  $(\mathcal{A}\text{-}\mathbf{Sch})$  of  $\mathcal{A}$ -schemes, namely the smallest complete full subcategory of  $(\mathcal{A}\text{-}\mathbf{Sch})$  containing  $(\mathcal{Q}\text{-}\mathbf{Sch})$ . In section 4, we will give a proof of the original version of Nagata embedding, which says that any separated scheme of finite type can be embedded as an open subscheme of a proper scheme.

**Notation and conventions.** In this paper, the algebraic type  $\mathcal{A}$  is always that of rings when we talk of  $\mathcal{A}$ -schemes. When we say ordinary schemes, we treat only coherent schemes and quasi-compact morphisms between them; to emphasize this assumption and to distinguish ordinary coherent schemes from  $\mathcal{A}$ -schemes, we will say  $\mathcal{Q}$ -schemes instead of coherent schemes.

For an  $\mathcal{A}$ -scheme  $X$ , the description  $|X|$  stands for the underlying topological space, which is coherent.

An  $\mathcal{A}$ -scheme  $X$  will be called *integral* if it is irreducible and reduced. This condition is in fact, stronger than assuming any section ring  $\mathcal{O}_X(U)$  is

integral.

A morphism of  $\mathcal{A}$ -schemes is *proper*, if it is separated and universally closed. We *do not include the condition “of finite type”*.

An open covering of an  $\mathcal{A}$  scheme  $X$  is denoted by  $\{\amalg U_{ijk} \rightrightarrows \amalg U_i\}$ ; here,  $\{U_i\}$  is a quasi-compact open covering of  $X$ , and  $\{U_{ijk}\}$  is a quasi-compact open covering of  $U_i \cap U_j$  for each  $i, j$ . Hence, there are open immersions  $U_{ijk} \rightarrow U_i$  and  $U_{ijk} \rightarrow U_j$  for each  $i, j, k$ . Moreover, if  $X$  is a  $\mathcal{Q}$ -scheme, we usually take  $U_i$  and  $U_{ijk}$  as open affine subschemes of  $X$ .

## 1 A brief review of $\mathcal{A}$ -schemes

In this section, we will recall some terminologies and definitions in [T]. A good reference for general lattice theories is [S].

A topological space  $X$  is *coherent*, if it is sober, quasi-compact, quasi-separated, and has a quasi-compact open basis.

The category (**Coh**) of coherent spaces and quasi-compact morphisms is isomorphic to the opposite category  $(\mathbf{DLat})^{\text{op}}$  of distributive lattices by the functor  $C(-)_{\text{cpt}}$ ; for a coherent space  $X$ , we may regard  $C(X)_{\text{cpt}}$  as the set of quasi-compact open subsets of  $X$ , or the set of their complements. A quasi-compact morphism  $f : X \rightarrow Y$  induces a morphism  $f^{-1} : C(Y)_{\text{cpt}} \rightarrow C(X)_{\text{cpt}}$  of lattices.

Therefore, for any algebraic type  $\mathcal{A}$ , we can regard an  $\mathcal{A}$ -valued sheaf on a coherent space  $X$  as a continuous covariant functor  $C(X)_{\text{cpt}} \rightarrow \mathcal{A}$ .

On a coherent space  $X$ , there is a canonical  $(\mathbf{DLat})$ -valued sheaf  $\tau_X$  on  $X$ , which is defined by  $U \mapsto C(U)_{\text{cpt}}$  for quasi-compact open  $U$ ; this extends uniquely to the entire Zariski site of  $X$ .

We have a functor  $\alpha_1 : (\mathbf{Rng}) \rightarrow (\mathbf{Dlat})$  from the category of commutative rings, which sends a ring  $R$  to the set of finitely generated ideals of  $R$  modulo the relation  $I^2 = I$ . Note that this gives the usual spectrum of rings, when combined with the previous isomorphism  $C(-)_{\text{cpt}}$ .

Also, we have a natural homomorphism  $R \rightarrow \alpha_1(R)$  of multiplicative monoids, sending  $a \in R$  to the principal ideal generated by  $a$ . This homomorphism commutes with localizations.

An  $\mathcal{A}$ -scheme is a triple  $X = (|X|, \mathcal{O}_X, \beta_X)$ , where

- (i)  $|X|$  is a coherent space (the “underlying space”),
- (ii)  $\mathcal{O}_X$  is a ring-valued sheaf on  $|X|$  (the “structure sheaf”), and

(iii)  $\beta_X : \alpha_1 \mathcal{O}_X \rightarrow \tau_X$  is a morphism of (**DLat**)-valued sheaves (the “support morphism”),

which satisfies the following condition: for an inclusion  $V \hookrightarrow U$  of quasi-compact open subsets of  $|X|$ , the restriction map  $\mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$  factors through  $\mathcal{O}_X(U)_Z$ , where  $\mathcal{O}_X(U)_Z$  is the localization along the multiplicative system

$$\{a \in \mathcal{O}_X(U) \mid \beta_X(a) \subset Z\},$$

where  $Z = U \setminus V$  is the complement closed subset of  $V$  in  $U$ . By this property,  $\mathcal{A}$ -schemes are locally ringed spaces. A morphism of  $\mathcal{A}$ -schemes  $f = (f, f^\#) : X \rightarrow Y$  is a morphism of ringed spaces, which commutes with the support morphism:

$$\begin{array}{ccc} \alpha_1 \mathcal{O}_Y & \xrightarrow{\alpha_1 f^\#} & f_* \alpha_1 \mathcal{O}_X \\ \beta_Y \downarrow & & \downarrow \beta_X \\ \tau_Y & \xrightarrow{f^{-1}} & f_* \tau_X \end{array}$$

The category ( **$\mathcal{A}$ -Sch**) of  $\mathcal{A}$ -schemes is complete, and co-complete.

We have a fully faithful functor ( **$\mathcal{Q}$ -Sch**)  $\rightarrow$  ( **$\mathcal{A}$ -Sch**), which preserves pull backs and finite patchings by quasi-compact opens.

Let  $\text{Spec } K \rightarrow S$  be a dominant morphism of  $\mathcal{A}$ -schemes, where  $S$  is integral and  $K$  is a field. The *Zariski-Riemann space*  $\text{ZR}^f(K, S)$  is a proper  $\mathcal{A}$ -scheme over  $S$ , defined as follows: the points of the underlying space  $|\text{ZR}^f(K, S)|$  corresponds to the set of dominant morphisms  $\mathcal{O}_{S,s} \rightarrow R$  of local rings, where  $R$ 's are valuation rings of  $K$ , and  $\mathcal{O}_{S,s}$ ' are local rings of  $S$ . The map  $|\text{ZR}^f(K, S)| \rightarrow |S|$  is defined naturally. The topology of  $|\text{ZR}^f(K, S)|$  is generated by the *domains*  $\{R \in |\text{ZR}^f(K, S)| \mid a \in R\}$  for  $a \in K \setminus \{0\}$  and the inverse images of the open subsets of  $S$ . For a quasi-compact open subset  $U$  of  $|\text{ZR}^f(K, S)|$ ,  $\mathcal{O}_{\text{ZR}^f(K, S)}(U)$  is the ring which is the intersection of all the valuation rings corresponding to the points in  $U$ .

For arbitrary dominant morphism  $X \rightarrow S$  of integral  $\mathcal{A}$ -schemes,  $\text{ZR}^f(X, S)$  is defined by the pushout of  $X \leftarrow \text{ZR}^f(K, X) \rightarrow \text{ZR}^f(K, S)$ , where  $K = Q(X)$  is the rational function field of  $X$ .  $X$  is *of profinite type* (resp. *strongly of profinite type*) over  $S$ , if the canonical map  $\text{ZR}^f(K, X) \rightarrow \text{ZR}^f(K, S)$  is a scheme-theoretic immersion (resp. open immersion).

Here, we would like to list up some important properties which is going to be used in the sequel:

- (1) A separated, of finite type morphism of  $\mathcal{Q}$ -schemes is always strongly of profinite type.
- (2) The map  $X \mapsto \mathrm{ZR}^f(X, S)$  gives the universal proper  $\mathcal{A}$ -scheme of profinite type over  $S$  for each integral  $\mathcal{A}$ -scheme  $X$  of profinite type over  $S$ .

## 2 Normalization

In this section, we fix an integral base  $\mathcal{A}$ -scheme  $S$ , and any  $\mathcal{A}$ -scheme is integral, of profinite type over  $S$ . We denote by **(Int.  $\mathcal{A}$ -Sch)** the category of integral  $\mathcal{A}$ -schemes of profinite type over  $S$  and dominant morphisms.

**Definition 2.1.** An  $\mathcal{A}$ -scheme  $X$  is *normal*, if the ring of every stalk  $\mathcal{O}_{X,x}$  is integrally closed.

**Remark 2.2.** We do not assume Noetherian property on normal rings (or schemes) in this paper.

**Theorem 2.3.** Let **(N.  $\mathcal{A}$ -Sch)** be the full subcategory of **(Int.  $\mathcal{A}$ -Sch)**, consisting of normal schemes, and  $U : \mathbf{(N. \mathcal{A}\text{-Sch})} \rightarrow \mathbf{(Int. \mathcal{A}\text{-Sch})}$  be the underlying functor. Then,  $U$  has a right adjoint ‘nor’. Moreover, the counit  $\eta : U \circ \mathrm{nor} \Rightarrow \mathrm{Id}$  is proper dominant.

We will refer to this right adjoint as the *normalization functor*.

*Proof.* The proof is somewhat long, so we will divide it into several steps. The construction of the normalization functor is analogous to that of Zariski-Riemann spaces, described in detail in [T]. We will denote by  $R^{\mathrm{nor}}$  the integral closure of a given integral domain  $R$  in the sequel.

Step 1: First, we will construct the underlying space of the normalization of a given integral  $\mathcal{A}$ -scheme  $X$ . Let  $\mathcal{N}_0^X$  be the set of finite sets of pairs  $(U, \alpha)$ , where

- (a)  $U$  is a quasi-compact open subset of  $X$ , and
- (b)  $\alpha \in \mathcal{O}_X(U)^{\mathrm{nor}} \setminus \{0\}$ .

Let  $\mathbf{a} = \{(U_i, \alpha_i)\}_i$ ,  $\mathbf{b} = \{(V_j, \beta_j)\}_j$  be two elements of  $\mathcal{N}_0^X$ . We define two operations  $+$ ,  $\cdot$  on  $\mathcal{N}_0^X$  by

$$\mathbf{a} + \mathbf{b} = \mathbf{a} \cup \mathbf{b}, \mathbf{a} \cdot \mathbf{b} = \{(U_i \cap V_j, \alpha_i \beta_j)\}_{ij}$$

For a pair  $(U, \alpha)$ , define  $U[\alpha]$  as

$$U[\alpha] = \{x \in U \mid x \text{ is in the image of } \text{Spec } \mathcal{O}_{X,x}[\alpha^{-1}] \rightarrow \text{Spec } \mathcal{O}_{X,x}\}.$$

For two elements  $\mathbf{a} = \{(U_i, \alpha_i)\}_i$ ,  $\mathbf{b} = \{(V_j, \beta_j)\}_j$ , the relation  $\mathbf{a} \prec \mathbf{b}$  holds if

- (a)  $U_i[\alpha_i] \subset \cup_j V_j[\beta_j]$  for any  $i$ , and
- (b) For any  $x \in U_i[\alpha_i]$ , set  $J_x = \{j \mid x \in V_j[\beta_j]\}$ . Then  $(\beta_j)_{j \in J_x}$  generates the unit ideal in  $\mathcal{O}_{X,x}^{\text{nor}}[\alpha_i^{-1}]$ .

Let  $\approx$  be the equivalence relation generated by  $\prec$ , and set  $\mathcal{N}^X = \mathcal{N}_0^X / \approx$ . The addition and multiplication of  $\mathcal{N}_0^X$  descends to  $\mathcal{N}^X$ , which makes  $\mathcal{N}^X$  into a distributive lattice. Set  $|X^{\text{nor}}| = \text{Spec } \mathcal{N}^X$ . This is the underlying space of the normalization  $X^{\text{nor}}$ .

Step 2: There is a natural homomorphism  $C(X)_{\text{cpt}} \rightarrow \mathcal{N}^X$  of distributive lattices, defined by  $Z \mapsto \{(Z, 1)\}$ . This defines a quasi-compact morphism  $\pi : |X^{\text{nor}}| \rightarrow |X|$  of coherent spaces.

Step 3: Let  $p$  be a point of  $|X^{\text{nor}}|$ , and set  $x = \pi(p)$ . Then,

$$\mathfrak{p} = \{a \in \mathcal{O}_{X,x}^{\text{nor}} \mid (X, a) \leq p\}$$

becomes a prime ideal of  $\mathcal{O}_{X,x}^{\text{nor}}$ . Let  $R_p$  be the localization of  $\mathcal{O}_{X,x}^{\text{nor}}$  by  $\mathfrak{p}$ . Then,  $R_p$  dominates  $\mathcal{O}_{X,x}$ .

Step 4: The structure sheaf  $\mathcal{O}_{X^{\text{nor}}}$  is defined by

$$U \mapsto \{a \in K \mid a \in R_p \text{ } (p \in U)\},$$

where  $K$  is the function field of  $X$ . The support morphism  $\beta_{X^{\text{nor}}} : \alpha_1 \mathcal{O}_{X^{\text{nor}}} \rightarrow \tau_{X^{\text{nor}}}$  is defined by

$$\alpha_1 \mathcal{O}_{X^{\text{nor}}}(U) \ni (a_1, \dots, a_n) \mapsto \{(U, a_i)\}_i.$$

This defines an  $\mathcal{A}$ -scheme  $X^{\text{nor}} = (|X^{\text{nor}}|, \mathcal{O}_{X^{\text{nor}}}, \beta_{X^{\text{nor}}})$ .

Step 5: We have a canonical morphism of sheaves  $\mathcal{O}_X \rightarrow \pi_* \mathcal{O}_{X^{\text{nor}}}$ , defined by the identity  $a \mapsto a$ . This yields a morphism  $\pi : X^{\text{nor}} \rightarrow X$  of  $\mathcal{A}$ -schemes. It is of profinite type, by the criterion 4.3.3 in [T].

Step 6: Let us show that  $\pi$  is proper.

We can see from the construction that we have a natural morphism  $\text{ZR}^f(K, X) \rightarrow X^{\text{nor}}$ : the morphism  $|\text{ZR}^f(K, X)| \rightarrow |X^{\text{nor}}|$  of underlying spaces is defined by

$$\mathcal{N}^X \rightarrow \mathcal{M}^X \quad (\{(U_i, \alpha_i)\}_i \mapsto \{(X \setminus U_i, \{\alpha_i^{-1}\})\}_i),$$

where  $\mathcal{M}^X = C(\text{ZR}^f(K, X))_{\text{cpt}}$ , and the morphism between the structure sheaves is canonical. Note that  $\text{ZR}^f(K, X)$  is already normal. This shows that  $X^{\text{nor}}$  is proper over  $X$  by the valuative criterion.

Step 7: We will show that the normalization is a functor. Let  $f : X \rightarrow Y$  be a dominant morphism of  $\mathcal{A}$ -schemes.  $|f^{\text{nor}}| : X^{\text{nor}} \rightarrow Y^{\text{nor}}$  is defined by

$$\mathcal{N}^Y \rightarrow \mathcal{N}^X : \{(U_i, \alpha_i)\}_i \mapsto \{(f^{-1}U_i, f^\# \alpha_i)\}_i.$$

The morphism  $f^\# : \mathcal{O}_Y \rightarrow f_* \mathcal{O}_X$  extends canonically to  $f^{\text{nor}, \#} : \mathcal{O}_{Y^{\text{nor}}} \rightarrow f_*^{\text{nor}} \mathcal{O}_{X^{\text{nor}}}$ . This gives a functor  $\text{nor} : (\mathbf{Int.} \ \mathcal{A}\text{-Sch}) \rightarrow (\mathbf{N.} \ \mathcal{A}\text{-Sch})$ .

Step 8: It remains to show that the normalization functor is indeed the right adjoint of the underlying functor. The unit  $\epsilon : \text{Id} \Rightarrow \text{nor} \circ U$  is the identity, since the normalization of a normal  $\mathcal{A}$ -scheme is trivial. The counit  $\eta : U \circ \text{nor} \Rightarrow \text{Id}$  is given by  $\pi$  defined above.

□

**Remark 2.4.** We know that the integral closure  $\tilde{R}$  of a Noetherian domain  $R$  is not Noetherian [N2]. Therefore, we must drop the ‘of finite type’ condition from the definition of properness, if we wish to say that “the normalization  $\text{Spec } \tilde{R} \rightarrow \text{Spec } R$  is proper”.

**Lemma 2.5.** Let  $X$  be a normal  $\mathcal{A}$ -scheme. Then.  $\mathcal{O}_X(U)$  is normal for any open  $U$ .

*Proof.* Let  $b \in K$  be an element which is integral over  $\mathcal{O}_X(U)$ , where  $K$  is the function field of  $X$ . Since  $b_x$  is integral over the stalk  $\mathcal{O}_{X,x}$  for any  $x \in U$  and  $\mathcal{O}_{X,x}$  is integrally closed, we have  $b_x \in \mathcal{O}_{X,x}$ . Hence,  $b \in \mathcal{O}_X(U)$ . □

**Proposition 2.6.** The normalization functor coincides with the usual normalization, when restricted to  $\mathcal{Q}$ -schemes.

*Proof.* First, we will show for affine schemes  $X = \operatorname{Spec} A$ . The universality of the normalization functor gives a canonical morphism  $f : \operatorname{Spec}(A^{\operatorname{nor}}) \rightarrow X^{\operatorname{nor}}$ . Since  $\Gamma(X^{\operatorname{nor}}, \mathcal{O}_{X,x})$  is normal, we have a canonical homomorphism  $A^{\operatorname{nor}} \rightarrow \Gamma(X^{\operatorname{nor}}, \mathcal{O}_{X,x})$ . This yields a morphism  $g : X^{\operatorname{nor}} \rightarrow \operatorname{Spec}(A^{\operatorname{nor}})$ . It is easy to check that these two morphisms  $f, g$  are inverse to each other.

It is obvious from the construction that normalization commutes with localizations. This shows that the normalization of any  $\mathcal{Q}$ -scheme coincides with the usual definition.  $\square$

### 3 Approximations by ordinary schemes

We fix an integral base  $\mathcal{Q}$ -scheme  $S$  in the sequel. The next proposition is pure category-theoretical and easy, so we will omit the proof.

**Proposition 3.1.** Let  $\mathcal{B}, \mathcal{C}$  be two categories, with  $\mathcal{B}$  finite complete and  $\mathcal{C}$  small complete. Let  $F : \mathcal{B} \rightarrow \mathcal{C}$  be a finite continuous functor, namely  $F$  preserves fiber products. For any object  $a$  of  $\mathcal{C}$ , The followings are equivalent:

- (i)  $a$  is isomorphic to a limit of the objects in  $\operatorname{Im} F$ .
- (ii)  $a$  is isomorphic to a filtered limit of the objects in  $\operatorname{Im} F$ .

**Definition 3.2.** Let  $X$  be an  $\mathcal{A}$ -scheme, and  $\mathcal{P}$  be a class of  $\mathcal{Q}$ -schemes.

- (1)  $X$  is *approximable by  $\mathcal{P}$* , if  $X$  is isomorphic to a filtered limit of some objects of  $\mathcal{P}$ .
- (2)  $X$  is *approximable*, if  $X$  is isomorphic to a filtered limit of some  $\mathcal{Q}$ -schemes.

**Proposition 3.3.** Any approximable  $\mathcal{A}$ -scheme is approximable by  $\mathcal{Q}$ -schemes of finite type.

*Proof.* It suffices to show that any  $\mathcal{Q}$ -scheme is approximable by  $\mathcal{Q}$ -schemes of finite type.

Let  $X$  be any  $\mathcal{Q}$ -scheme, and  $\{\coprod U_{ijk} \rightarrow \coprod U_i\}$  be a finite affine covering of  $X$ . Since  $U_{ijk} \rightarrow U_i$  is quasi-compact,  $U_{ijk}$  is of finite type over  $U_i$ . Thus, we



have approximations  $U_i = \varprojlim_{\lambda} U_i^{\lambda}$  and  $U_{ijk} = \varprojlim_{\lambda} U_{ijk}^{\lambda}$  so that  $U_i^{\lambda}$  and  $U_{ijk}^{\lambda}$  are of finite type and  $U_{ijk}^{\lambda} \rightarrow U_i^{\lambda}$  are open immersions. We may also assume that the above limits are filtered. Since filtered limits and finite colimits commute, we have

$$X = \varinjlim_i U_i = \varinjlim_i \varprojlim_{\lambda} U_i^{\lambda} = \varprojlim_{\lambda} \varinjlim_i U_i^{\lambda}$$

and  $\varinjlim_i U_i^{\lambda}$  is a  $\mathcal{Q}$ -scheme of finite type.  $\square$

**Definition 3.4.** Let  $X, Y$  be two integral  $\mathcal{A}$ -schemes. A morphism  $f : X \rightarrow Y$  is *birational*, if  $f$  induces an isomorphism  $Q(X) \simeq Q(Y)$  between the rational function fields.

**Remark 3.5.** Note that, the morphism being birational does not imply the existence of an open dense subset  $U$  of  $X$  such that  $U \simeq f(U)$ .

**Proposition 3.6.** Let  $X$  be an approximable  $\mathcal{A}$ -scheme, say  $X = \varprojlim_{\lambda} X^{\lambda}$  where  $X^{\lambda}$ 's are  $\mathcal{Q}$ -schemes.

- (1) If  $X$  is reduced, then  $X$  is approximable by reduced  $\mathcal{Q}$ -schemes.
- (2) If  $X$  is integral, then  $X$  is approximable by integral  $\mathcal{Q}$ -schemes.
- (3) Further, if the rational function field  $Q(X)$  is finitely generated over an integral base  $\mathcal{Q}$ -scheme, then  $X$  is approximable by integral  $\mathcal{Q}$ -schemes birational to  $X$ .
- (4) If  $X$  is normal, then  $X$  is approximable by normal  $\mathcal{Q}$ -schemes.
- (5) If  $X$  is proper and approximable by separated  $\mathcal{Q}$ -schemes, then  $X$  is approximable by proper (and of finite type)  $\mathcal{Q}$ -schemes.

*Proof.* The proofs are all similar, so let us just see (1).

Since  $X$  is reduced,  $X \rightarrow X^{\lambda}$  factors through the reduced  $\mathcal{Q}$ -scheme  $(X^{\lambda})_{\text{red}}$ . This shows that  $X \simeq \varprojlim_{\lambda} (X^{\lambda})_{\text{red}}$ .  $\square$

**Proposition 3.7.** Let  $f : X \rightarrow Y$  be a morphism of  $\mathcal{A}$ -schemes over  $S$ , with  $X$  approximable and  $Y$  a  $\mathcal{Q}$ -scheme, of finite type over  $S$ .

- (1) Suppose  $X$  is a filtered projective limit  $\varprojlim_{\lambda} X^{\lambda}$  of  $\mathcal{Q}$ -schemes. Then,  $f$  factors through  $X \rightarrow X^{\lambda}$  for some  $\lambda$ .

- (2) Furthermore, if  $X$  is proper over  $S$  and approximable by separated  $\mathcal{Q}$ -schemes, and  $Y$  is separated over  $S$ , then the above  $X^\lambda$  can be chosen to be a proper scheme over  $Y$ .

*Proof.* (1) We may assume that  $Y$  is affine. Since  $Y$  is of finite type and  $\Gamma(X, \mathcal{O}_X)$  is a filtered colimit of  $\Gamma(X_\lambda, \mathcal{O}_{X^\lambda})$ ,  $f$  factors through  $X^\lambda$  for some  $\lambda$ .

- (2) By the above proposition, we may assume that  $X^\lambda$ 's are proper over the base scheme  $S$ . Since  $Y$  is separated, the morphism  $X^\lambda \rightarrow Y$  is proper.

□

**Theorem 3.8.** Let  $f : X \rightarrow Y$  be a proper birational morphism, where  $X$  is an integral  $\mathcal{A}$ -scheme approximable by separated  $\mathcal{Q}$ -schemes, and  $Y$  a normal  $\mathcal{Q}$ -scheme separated and of finite type over  $S$ . Then,  $f_*\mathcal{O}_X = \mathcal{O}_Y$ .

*Proof.* The previous proposition shows that  $f$  factors through proper morphisms  $f_\lambda : X^\lambda \rightarrow Y$ , where  $X = \varprojlim_\lambda X^\lambda$  and  $\{X^\lambda\}$  is a filtered system of integral  $\mathcal{Q}$ -schemes, proper birational and of finite type over  $Y$ . Since  $Y$  is normal, the usual Zariski's main theorem tells that  $\mathcal{O}_Y \rightarrow (f_\lambda)_*\mathcal{O}_{X^\lambda}$  is an isomorphism (Corollary III. 11.4 of [H]), and  $f_*\mathcal{O}_X$  coincides with the right hand side, since it is a colimit of  $(f_\lambda)_*\mathcal{O}_{X^\lambda}$ 's. □

Since “of profinite type” morphisms are stable under taking limits, approximable  $\mathcal{A}$ -schemes are necessarily of profinite type over  $S$ .

**Theorem 3.9.** Let  $X$  be a normal  $\mathcal{A}$ -scheme, proper and of profinite type over the integral base  $\mathcal{Q}$ -scheme  $S$ . Assume that the rational function field  $Q(X)$  is finitely generated over  $Q(S)$ . The followings are equivalent:

- (i)  $X$  is approximable by separated  $\mathcal{Q}$ -schemes.
- (ii) Let  $\mathcal{U} = \{\coprod U_{ijk} \rightrightarrows \coprod U_i\}$  be any quasi-compact open covering of  $X$ . Then, there exists a refinement  $\coprod V_{ijk} \rightrightarrows \coprod V_i$  of  $\mathcal{U}$  such that  $\mathrm{Spec} \mathcal{O}_X(V_{ijk}) \rightarrow \mathrm{Spec} \mathcal{O}_X(V_i)$  are open immersions.

*Proof.* (i) $\Rightarrow$ (ii):  $X$  can be written as a filtered limit  $X = \varprojlim_\lambda X^\lambda$ , where  $X^\lambda$ 's are normal  $\mathcal{Q}$ -schemes, proper and of finite type over  $S$ . Since the number of  $U_i$ 's and  $U_{ijk}$ 's are finite,  $U_i = \pi^{-1}\tilde{U}_i$ ,  $U_{ijk} = \pi^{-1}\tilde{U}_{ijk}$  for some  $\pi : X \rightarrow X^\lambda$ , where  $\tilde{U}_i$ 's and  $\tilde{U}_{ijk}$ 's are quasi-compact open subsets of  $X^\lambda$ .

Take any refinement  $\{\Pi\tilde{V}_{ijk} \rightarrow \Pi\tilde{V}_i\}$  of  $\tilde{\mathcal{U}} = \{\Pi\tilde{U}_{ijk} \rightarrow \Pi\tilde{U}_i\}$ , by affine opens  $\tilde{V}_{ijk}$  and  $\tilde{V}_i$ . Set  $V_{ijk} = \pi^{-1}\tilde{V}_{ijk}$  and  $V_i = \pi^{-1}\tilde{V}_i$ . Since  $V_i \rightarrow \tilde{V}_i$  is proper and  $\tilde{V}_i$  is normal, of finite type, Theorem 3.8 implies that  $\mathcal{O}_X(V_i) = \mathcal{O}_{X^\lambda}(\tilde{V}_i)$ . This shows that  $\text{Spec } \mathcal{O}_X(V_{ijk}) \rightarrow \text{Spec } \mathcal{O}_X(V_i)$  are open immersions.

(ii) $\Rightarrow$ (i): For any covering  $\mathcal{U} = \{\Pi U_{ijk} \rightrightarrows \Pi U_i\}$  of  $X$ , the refinement  $\Pi V_{ijk} \rightrightarrows \Pi V_i$  gives open immersions  $\text{Spec } \mathcal{O}_X(V_{ijk}) \rightarrow \text{Spec } \mathcal{O}_X(V_i)$  which patches up to give a  $\mathcal{Q}$ -scheme  $X(\mathcal{U})$  and the canonical morphism  $\pi_{\mathcal{U}} : X \rightarrow X(\mathcal{U})$ . The covering  $\mathcal{U}$  is a pull back of a covering of  $X(\mathcal{U})$ , and ditto for the elements of  $\mathcal{O}_X(U_i)$ 's. From this observation, we see that the induced morphism  $X \rightarrow \varprojlim_{\mathcal{U}} X(\mathcal{U})$  is an isomorphism. It is clear from the construction that  $X(\mathcal{U})$  is proper.  $\square$

## 4 Another proof of Nagata embedding

In the sequel, any  $\mathcal{A}$ -schemes are integral.

**Definition 4.1.** Let  $S$  be a  $\mathcal{Q}$ -scheme, and  $X$  be a  $\mathcal{Q}$ -scheme over  $S$ . We say that  $X$  is *compactifiable* over  $S$ , if there is an open immersion  $X \rightarrow Y$  where  $Y$  is a  $\mathcal{Q}$ -scheme, proper, of finite type over  $S$ .

**Proposition 4.2.** Let  $S$  be a  $\mathcal{Q}$ -scheme, and  $X$  be a  $\mathcal{Q}$ -scheme over  $S$ . The followings are equivalent:

- (i)  $X$  is compactifiable over  $S$ .
- (ii)  $\text{ZR}^f(X, S)$  is approximable by separated  $\mathcal{Q}$ -schemes, and the natural map  $X \rightarrow \text{ZR}^f(X, S)$  is an open immersion.

*Proof.* (i) $\Rightarrow$ (ii): There exists an open immersion  $X \rightarrow Y$  into a  $\mathcal{Q}$ -scheme  $Y$ , proper of finite type over  $S$ . This morphism factors through  $\text{ZR}^f(X, S)$  by the universal property. We will show that for any quasi-compact open subset  $U$  of  $\text{ZR}^f(X, S)$ , there exists a proper birational morphism  $Y' \rightarrow Y$ , such that  $g^{-1}(V) = U$  for some quasi-compact open subset  $V$  of  $Y'$ , where  $g : \text{ZR}^f(X, S) \rightarrow Y'$  is the canonical extension of  $f : X \rightarrow Y$ :

$$\begin{array}{ccccc}
 U & \longrightarrow & \text{ZR}^f(X, S) & \longleftarrow & X \\
 \vdots & & \square & & \downarrow \\
 V & \dashrightarrow & Y' & \dashrightarrow & Y
 \end{array}$$

(The diagram shows a commutative square with a dashed arrow from  $U$  to  $V$  and a dashed arrow from  $\text{ZR}^f(X, S)$  to  $Y'$ . A solid arrow  $g$  points from  $\text{ZR}^f(X, S)$  to  $Y'$ , and a solid arrow  $f$  points from  $X$  to  $Y$ . A solid arrow points from  $U$  to  $\text{ZR}^f(X, S)$ , and a solid arrow points from  $X$  to  $\text{ZR}^f(X, S)$ . A solid arrow points from  $V$  to  $Y'$ , and a solid arrow points from  $Y'$  to  $Y$ .)

By the construction of  $\mathrm{ZR}^f(X, S)$ , we may assume  $U$  is of the form  $U(W, \alpha)$ , where  $W$  is a quasi-compact open subset of  $S$  and  $\alpha$  is a finite subset of  $K \setminus \{0\}$ , and

$$U(W, \alpha) = \pi^{-1}(W) \cap \{p \in \mathrm{ZR}^f(X, S) \mid \alpha \subset \mathcal{O}_{\mathrm{ZR}^f(X, S), p}\}.$$

Note that  $f(U \cap X)$  is open in  $Y$ , since  $X \rightarrow \mathrm{ZR}^f(X, S)$  is an open immersion. Suppose  $\alpha = \{a_i/b_i\}_i$ , where  $a_i, b_i \in \mathcal{O}_Y$  locally. Let  $Y' \rightarrow Y$  be the blow up along  $(Y \setminus X) \cap \mathrm{Supp}(a_i, b_i)$ . Then, either  $a_i/b_i$  or  $b_i/a_i$  is in  $\mathcal{O}_{Y'}$  locally, which shows that the domain of  $a_i/b_i$  is open in  $Y'$ . This shows that  $U$  is the pull back of some  $V$  by the morphism  $g : \mathrm{ZR}^f(X, S) \rightarrow Y'$ . Hence,  $\mathrm{ZR}^f(X, S) \rightarrow \varprojlim_{\lambda} Y^{\lambda}$  becomes a homeomorphism on the underlying space, where  $Y^{\infty} = \varprojlim_{\lambda} Y^{\lambda}$  is the filtered projective limit of  $X$ -admissible blow-ups of  $Y$ . A similar argument shows that  $\mathcal{O}_{Y^{\infty}} \rightarrow \mathcal{O}_{\mathrm{ZR}^f(X, S)}$  also becomes an isomorphism. Note that  $Y^{\lambda}$ 's are separated over  $S$ , since we only used blow-ups.

(ii) $\Rightarrow$ (i): The Zariski-Riemann space  $\mathrm{ZR}^f(X, S)$  can be written as a form  $\varprojlim_{\lambda} Y^{\lambda}$ , where  $Y^{\lambda}$ 's are proper, of finite type  $\mathcal{Q}$ -schemes. Since  $X \rightarrow \mathrm{ZR}^f(X, S)$  is an open immersion and  $X$  is quasi-compact,  $X \rightarrow \mathrm{ZR}^f(X, S) \rightarrow Y^{\lambda}$  becomes an open immersion for some  $\lambda$ .  $\square$

Now, we are on the stage to give the proof of the Nagata embedding.

**Theorem 4.3** (Nagata). Let  $S$  be a  $\mathcal{Q}$ -scheme, and  $X$  be a  $\mathcal{Q}$ -scheme, separated and of finite type over  $S$ . Then,  $X$  is compactifiable over  $S$ .

In this section, we will prove this theorem for the essential case, namely when  $S$  and  $X$  are integral. This restriction is due to the fact that we simply haven't established the theorem of Zariski-Riemann spaces for non-integral schemes.

Since  $X$  is quasi-compact, and affine schemes of finite type over  $S$  is obviously compactifiable, it suffices to prove the following proposition:

**Proposition 4.4.** Let  $V_1$  and  $V_2$  be compactifiable open sub- $\mathcal{Q}$ -schemes of a  $\mathcal{Q}$ -scheme  $X$  separated over  $S$ , with  $X = V_1 \cup V_2$ . Then  $X$  is also compactifiable.

*Proof.* Consider  $\mathrm{ZR}^f(X, S)$ . Since  $X$  is separated, of finite type over  $S$ , the morphism  $X \rightarrow \mathrm{ZR}^f(X, S)$  is an open immersion by Corollary 4.4.6 of [T].

Let  $W_1$  (resp.  $W_2$ ) be the complement of the closure of  $V_2 \setminus V_1$  (resp.  $V_1 \setminus V_2$ ) in  $\mathrm{ZR}^f(X, S)$ .

We can see that  $W_1 \cap W_2 = V_1 \cap V_2$ , since the interior of the complement of  $V_1 \cup V_2$  is empty. Next, we see that  $W_1 \cup W_2 = \mathrm{ZR}^f(X, S)$ . For this, it suffices to show that  $\overline{V_2 \setminus V_1} \cap \overline{V_1 \setminus V_2} = \emptyset$ . Suppose there is a point  $p$  in  $\overline{V_2 \setminus V_1} \cap \overline{V_1 \setminus V_2}$ . Since  $V_2 \setminus V_1$  and  $V_1 \setminus V_2$  are coherent subsets of  $\mathrm{ZR}^f(X, S)$ ,  $p$  must be a specialization of some  $x_1 \in V_2 \setminus V_1$  and  $x_2 \in V_1 \setminus V_2$  by Corollary 1.2.8 of [T]. Since  $\mathrm{ZR}^f(K, S) \rightarrow \mathrm{ZR}^f(X, S)$  is surjective, there are inverse images  $y_i \in \mathrm{ZR}^f(K, S)$  of  $x_i$  such that  $y_i$  specializes to  $p$ . The points in  $\mathrm{ZR}^f(K, S)$  are valuation rings, hence  $y_2$  must be the specialization of  $y_1$ , or the converse. In either cases, this contradicts to the fact that  $x_1$  and  $x_2$  has no specialization-generalization relations. This also shows that  $W_1$  and  $W_2$  are quasi-compact. The morphism  $p_1 : \mathrm{ZR}^f(V_1, S) \rightarrow \mathrm{ZR}^f(X, S)$  induces an isomorphism on  $W_1$ , hence  $W_1$  is approximable by  $\mathcal{Q}$ -morphisms of finite type over  $S$ , ditto for  $W_2$ .

Take any  $\mathcal{Q}$ -model  $Y_i$  of  $W_i$  (namely, a morphism  $\pi_i : W_i \rightarrow Y_i$  where  $Y_i$  is a  $\mathcal{Q}$ -scheme) such that the morphism  $\pi_i$  induces an isomorphism on  $V_i$ . Then,  $Y_1$  and  $Y_2$  can be patched along  $\pi_1(W_1 \cap W_2) \simeq \pi_2(W_1 \cap W_2)$  to obtain a  $\mathcal{Q}$ -scheme  $Y$  of finite type, and a surjective morphism  $\mathrm{ZR}^f(X, S) = W_1 \cup W_2 \rightarrow Y$ . This shows that  $Y$  is proper.  $\square$

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S. TAKAGI: DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KYOTO UNIVERSITY, KYOTO, 606-8502, JAPAN

*E-mail address:* takagi@math.kyoto-u.ac.jp